

Note on packing of edge-disjoint spanning trees in sparse random graphs*

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Abstract

The *spanning tree packing number* of a graph G is the maximum number of edge-disjoint spanning trees contained in G . Let $k \geq 1$ be a fixed integer. Palmer and Spencer proved that in almost every random graph process, the hitting time for having k edge-disjoint spanning trees equals the hitting time for having minimum degree k . In this paper, we prove that for any p such that $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$, almost surely the random graph $G(n, p)$ satisfies that the spanning tree packing number is equal to the minimum degree. Note that this bound for p will allow the minimum degree to be a function of n , and in this sense we improve the result of Palmer and Spencer. Moreover, we also obtain that for any p such that $p \geq (51 \log n)/n$, almost surely the random graph $G(n, p)$ satisfies that the spanning tree packing number is less than the minimum degree.

Keywords: edge-disjoint spanning trees, random graph, minimum degree

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1 Introduction

For a graph G of order n , the *spanning tree packing number*, denoted by $\sigma = \sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G . The spanning tree

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packing problem has long been one of the main motives in graph theory. In 1961, Nash-Williams [6] and Tutte [9] independently obtained a necessary and sufficient condition for a graph to have k edge-disjoint spanning trees.

Theorem 1.1 [6, 9] *A graph $G = G(V, E)$ contains k edge-disjoint spanning trees if and only if*

$$|E_G(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$$

for every partition \mathcal{P} of V , where $|\mathcal{P}|$ denotes the the number of sets in \mathcal{P} and $E_G(\mathcal{P})$ are the crossing edges of G , i.e., edges joining vertices that are in different sets of \mathcal{P} .

In the same papers, they also proved that $\sigma(G) = \lfloor \eta(G) \rfloor$, where $\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{\omega(G-E)-1}$.

Frieze and Luczak [5] firstly considered the spanning tree packing number of a random graph and they obtained that for a fixed integer $k \geq 2$ the random graph $G_{k-\text{out}}$ almost surely has k edge-disjoint spanning trees. Moreover, Palmer and Spencer [8] proved that in almost every random graph process, the hitting time for having k edge-disjoint spanning trees equals the hitting time for having minimum degree k , for any fixed positive integer k . In other words, considering the random graph $G(n, p)$, for any fixed positive integer k , if $p(n) \leq \frac{\log n + k \log \log n - \omega(1)}{n}$, the probability that the spanning tree packing number equals the minimum degree approaches to 1 as $n \rightarrow \infty$. Note that for a fixed k , $\frac{\log n + k \log \log n - \omega(1)}{n}$ is the best upper bound for $p(n)$ to guarantee $\delta(G(n, p)) \leq k$ a.s.

On the other hand, in Catlin's paper [4] it was found that if the edge probability is rather large, then almost surely the random graph $G(n, p)$ has $\sigma(G) = \lfloor |E(G)|/(n-1) \rfloor$, which is less than the minimum degree of G . We refer papers [4] and [7] to the reader for more details.

A natural question is whether there exists a largest $q(n)$ such that for every $p \leq q(n)$, almost surely the random graph $G(n, p)$ satisfies that the spanning tree packing number equals the minimum degree.

In this paper, we partly answer this question by establishing the following two theorems. The first theorem establishes a lower bound of $q(n)$ with $q(n) \geq (1.1 \log n)/n$. Note that this bound for p will allow the minimum degree to be a function of n , and in this sense we improve the result of Palmer and Spencer.

Theorem 1.2 *For any p such that $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$, almost surely the random graph $G \sim G(n, p)$ satisfies that the spanning tree packing number is equal to the minimum degree, i.e.*

$$\lim_{n \rightarrow \infty} \Pr(\sigma(G) = \delta(G)) = 1.$$

The second theorem gives an upper bound of $q(n)$ with $q(n) \leq (51 \log n)/n$.

Theorem 1.3 *For any p such that $p \geq (51 \log n)/n$, almost surely the random graph $G \sim G(n, p)$ satisfies that the spanning tree packing number is less than the minimum degree, i.e.*

$$\lim_{n \rightarrow \infty} \Pr(\sigma(G) < \delta(G)) = 1.$$

The rest of the paper is organized as follows: In Section 2, we list some basic notations and collect a few auxiliary results. Then we prove Theorem 1.2 in Section 3 and give the proof of Theorem 1.3 in Section 4.

2 Preliminaries

2.1 Notation

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are denoted by $|V(G)|$ and $|E(G)|$, respectively. Given a set $A \subseteq V(G)$, \bar{A} denotes the set $V(G) \setminus A$, and the subgraph of G induced by A is denoted by $G[A]$. For two disjoint sets $A, B \subseteq V(G)$, $E(A, B)$ denotes the set of edges between A and B . The minimum degree of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. For more notations we refer to the book [3].

In this paper, we consider the Erdős-Rényi random graph $G(n, p)$, which is a graph with n vertices where each of the $\binom{n}{2}$ potential edges appears with probability p , independently. Given a graph property \mathcal{Q} , we say that a random graph $G(n, p)$ has property \mathcal{Q} almost surely (a.s.), if the probability that the random graph $G(n, p)$ has the property \mathcal{Q} approaches to 1 as $n \rightarrow \infty$. Therefore, from now on and throughout the rest of this paper, when needed we always assume that n is large enough. For a positive integer n and $0 \leq p \leq 1$, we denote by $\text{Bin}(n, p)$ the binomial random variable with parameters n and p . $X \sim \text{Bin}(n, p)$ means that X and $\text{Bin}(n, p)$ have the same distribution. We always write \log for the natural logarithm.

In this paper, we use the following standard asymptotic notations: as $n \rightarrow \infty$, $f(n) = o(g(n))$ means that $f(n)/g(n) \rightarrow 0$; $f(n) = \omega(g(n))$ means that $f(n)/g(n) \rightarrow \infty$; $f(n) = O(g(n))$ means that there exists a constant C such that $|f(n)| \leq Cg(n)$; $f(n) = \Omega(g(n))$ means that there exists a constant $c > 0$ such that $f(n) \geq cg(n)$.

2.2 Inequalities

In our proofs, we often use the following inequalities [1].

Lemma 2.1 (*Chernoff's inequality*) Let n be a positive integer, $p \in [0, 1]$ and $X \sim \text{Bin}(n, p)$. For every positive a ,

$$\Pr(X < np - a) < \exp\left(-\frac{a^2}{2np}\right) \quad \text{and} \quad \Pr(X > np + a) < \exp\left(-\frac{a^2}{2np} + \frac{a^3}{2(np)^2}\right)$$

In particular, if $a \leq np/2$, then

$$\Pr(X > np + a) < \exp\left(-\frac{a^2}{4np}\right).$$

Lemma 2.2 For $1 \leq k \leq n$,

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

3 Proof of Theorem 1.2

In this section, we first give an upper bound of the minimum degree in Lemma 3.1, then we show that for any set $S \subseteq V(G)$, there are enough edges between S and \bar{S} in Lemmas 3.3, 3.4 and 3.5. Finally, we use these lemmas and Theorem 1.1 to prove Theorem 1.2.

Lemma 3.1 Let $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$ and $G \sim G(n, p)$. Then $\delta(G) \leq \log n/30$ a.s..

Proof. Let $k = \lfloor \log n/30 \rfloor$. It is obvious that if $\delta(G) \leq k$ a.s., then this is also true for every $p' \leq p$ due to monotonicity. Therefore, it is sufficient to prove that for $p = (1.1 \log n)/(n - k)$, $\delta(G) \leq k$ a.s.

Let v be an arbitrary vertex of G . We have

$$\begin{aligned} \Pr(\deg(v) = k) &= \Pr(\text{Bin}(n-1, p) = k) \\ &= \binom{n-1}{k} p^k (1-p)^{n-1-k} \\ &\geq \left(\frac{n-k}{k}\right)^k p^k (1-p)^{n-k} \\ &= (1-o(1)) \left(\frac{(n-k)p}{k}\right)^k e^{-p(n-k)} \\ &\geq (1-o(1)) (33)^{\log n/30} n^{-1.1} \\ &= \omega(1/n). \end{aligned}$$

Then we use a basic result in the theory of random graphs due to Bollobás (see e.g. [2], Chapter 3) which asserts that if $\Pr(\text{Bin}(n-1, p) = k) = \omega(1/n)$, then $\delta(G) \leq k$ a.s. This completes the proof. \blacksquare

A vertex is called *small* if its degree is less than or equal to $\log n/6$, and otherwise it is called *large*. Denote by SMALL and LARGE the set of all small vertices and all large vertices, respectively. Then we can obtain an important structural property of random graphs as follows.

Lemma 3.2 *If $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$, then a.s. the random graph $G \sim G(n, p)$ satisfies the following properties:*

- (1) $|SMALL| \leq n^{1/2}$;
- (2) No pair of small vertices are adjacent or share a common neighbor.

Proof. (1) Let $s = \lceil n^{1/2} \rceil$. Assume that there exists a vertex set S with order s such that each vertex $v \in S$ is small, which happens with probability at most

$$\begin{aligned}
& \binom{n}{s} \left(\sum_{k=0}^{\log n/6} \binom{n-1}{k} p^k (1-p)^{n-1-k} \right)^s \\
& \leq \left(\frac{ne}{s} \right)^s \left(\frac{\log n}{6} \left(\frac{6(n-1)e}{\log n} \right)^{\log n/6} p^{\log n/6} e^{-p(n-1-\log n/6)} \right)^s \\
& \leq \left(\frac{ne}{s} \cdot \frac{\log n}{6} \cdot (6.6e)^{\log n/6} \cdot e^{-\log n + p + (\log n/6)p} \right)^s \\
& \leq \left(\frac{ne}{s} \cdot \frac{\log n}{6} \cdot n^{0.482} \cdot n^{-1} \cdot O(1) \right)^s \\
& = O(n^{-0.01s}).
\end{aligned}$$

It means that a.s. $|SMALL| \leq n^{1/2}$.

- (2) The probability that G violates property (2) can be bounded as follows:

$$\begin{aligned}
\Pr(G \text{ violates (2)}) & \leq \binom{n}{2} \cdot p \cdot \left(\sum_{k=0}^{\log n/6} \binom{n-1}{k} p^k (1-p)^{n-1-k} \right)^2 \\
& \quad + \binom{n}{2} \cdot \binom{n}{1} \cdot p^2 \cdot \left(\sum_{k=0}^{\log n/6-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \right)^2 \\
& = O(n^{-0.01}),
\end{aligned}$$

which implies that property (2) holds a.s. \blacksquare

Lemma 3.3 *Let $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$ and $G \sim G(n, p)$. Then a.s. for any vertex subset S such that $\emptyset \neq S \subseteq \text{LARGE}$ and $|S| \leq n/(\log n)^3$, $|E(S, \bar{S})| \geq (\log n/10) \cdot |S|$.*

Proof. We prove this lemma by contradiction. Assume that there exists a vertex subset S such that $\emptyset \neq S \subseteq \text{LARGE}$, $|S| \leq n/(\log n)^3$ and $|E(S, \bar{S})| < (\log n/10) \cdot |S|$. Then the induced subgraph $G[S]$ contains $|S|$ vertices and at least $(\frac{\log n}{6}|S| - \frac{\log n}{10}|S|)/2 = \frac{\log n}{30}|S|$ edges. The probability for the existence of such S can be bounded as follows:

$$\begin{aligned}
& \Pr \left(\bigcup_{\substack{|S| \leq n/(\log n)^3 \\ S \subseteq \text{LARGE}}} (|E(S, \bar{S})| < (\log n/10) \cdot |S|) \right) \\
& \leq \sum_{r=1}^{\frac{n}{(\log n)^3}} \left(\binom{n}{r} \cdot \sum_{k=(\log n/30) \cdot r}^{\binom{r}{2}} \left(\binom{\binom{r}{2}}{k} p^k (1-p)^{\binom{r}{2}-k} \right) \right) \\
& \leq \sum_{r=1}^{\frac{n}{(\log n)^3}} \left(\left(\frac{ne}{r} \right)^r \cdot \binom{r}{2} \left(\frac{r^2}{2} \right) \left(\frac{\log n}{30} \right)^r p^{\frac{\log n}{30} r} (1-p)^{\binom{r}{2} - \frac{\log n}{30} r} \right) \\
& \leq \sum_{r=1}^{\frac{n}{(\log n)^3}} \left(\left(\frac{ne}{r} \right)^r \cdot \frac{r^2}{2} \left(\frac{15erp}{\log n} \right)^{\frac{\log n}{30} r} e^{-\frac{\log n}{2n} r^2 + \frac{(\log n)^2}{30n} r} \right) \\
& = O(n^{-20}),
\end{aligned}$$

which implies the correctness of the lemma. ■

Lemma 3.4 *Let $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$ and $G \sim G(n, p)$. Then a.s. for any vertex subset S such that $n/(\log n)^3 \leq |S| \leq n/2$, $|E(S, \bar{S})| \geq (\log n/10) \cdot |S|$.*

Proof. The Event that there exists a vertex subset S such that $n/(\log n)^3 \leq |S| \leq n/2$ and $|E(S, \bar{S})| < (\log n/10) \cdot |S|$ happens with probability at most

$$\begin{aligned}
& \sum_{|S|=n/(\log n)^3}^{n/2} \left(\binom{n}{s} \Pr(|E(S, \bar{S})| < \frac{\log n}{10} |S|) \right) \\
& \leq \sum_{|S|=n/(\log n)^3}^{n/2} \left(\left(\frac{ne}{s} \right)^s \cdot e^{-\frac{1}{2} \left(1 - \frac{\log n}{10p(n-s)}\right)^2 \cdot (n-s)sp} \right) \\
& \leq \sum_{|S|=n/(\log n)^3}^{n/2} \left(\left(\frac{ne}{s} \right)^s \cdot e^{-\frac{s \log n}{10}} \right) \\
& \leq \sum_{|S|=n/(\log n)^3}^{n/2} \left(\frac{n^{9/10} e}{s} \right)^s \\
& \leq \frac{n}{2} \left(\frac{e(\log n)^3}{n^{1/10}} \right)^{\frac{n}{(\log n)^3}} \\
& = o(n^{-20}),
\end{aligned}$$

which gives precisely what we want. ■

Lemma 3.5 *Let $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$ and $G \sim G(n, p)$. Then a.s. for any vertex subset S such that $1 \leq |S| \leq n/(\log n)^3$, $|E(S, \bar{S})| \geq \delta(G) \cdot |S|$.*

Proof. For any set $S \subseteq V(G)$ and $1 \leq |S| \leq n/(\log n)^3$. Let $S = S_1 \cup S_2$, where $S_1 \subseteq \text{LARGE}$, $S_2 \subseteq \text{SMALL}$. Then $|E(S, \bar{S})| = |E(S_1, \bar{S}_1)| + |E(S_2, \bar{S}_2)| - 2|E(S_1, S_2)|$. By Lemmas 3.1, 3.2 and 3.3, we get that $|E(S_2, \bar{S}_2)| \geq \delta(G) \cdot |S_2|$, $|E(S_1, S_2)| \leq |S_1|$ and $|E(S_1, \bar{S}_1)| \geq (\log n/10) \cdot |S_1|$, respectively. It follows that

$$|E(S, \bar{S})| \geq \left(\frac{\log n}{10} - 2 \right) \cdot |S_1| + \delta(G) \cdot |S_2| \geq \delta(G) \cdot |S|.$$

The proof is thus completed. ■

At present, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2: Recall that $(\log n + \omega(1))/n \leq p \leq (1.1 \log n)/n$. Consider the random graph $G \sim G(n, p)$. Obviously, $\sigma(G) \leq \delta(G)$ always holds. We only need to prove that a.s. $\sigma(G) \geq \delta(G)$. By Theorem 1.1, it is sufficient to show that for any partition \mathcal{P} of $V(G)$, $|E_{\mathcal{P}}(G)| \geq \delta(G) \cdot (|\mathcal{P}| - 1)$.

Given a partition $\mathcal{P} = \{V_1, V_2, \dots, V_t\}$ with $t \geq 2$. Suppose $|V_1| \geq |V_2| \geq \dots \geq |V_t|$. We distinguish two cases to prove the theorem, according to the order of V_1 .

Case 1. $|V_1| \geq \frac{n}{2}$.

Since $|V_1| \geq \frac{n}{2}$, then $|\bar{V}_1| \leq \frac{n}{2}$ and $|V_i| \leq \frac{n}{2}$ for $2 \leq i \leq t$. By Lemmas 3.1, 3.4 and 3.5, $|E(V_1, \bar{V}_1)| \geq \delta(G) \cdot |\bar{V}_1|$ and $|E(V_i, \bar{V}_i)| \geq \delta(G) \cdot |V_i|$ for $2 \leq i \leq t$. Therefore,

$$|E_{\mathcal{P}}(G)| = \frac{1}{2} \sum_{i=1}^t |E(V_i, \bar{V}_i)| \geq \frac{1}{2} \delta(G) \cdot |\bar{V}_1| + \frac{1}{2} \delta(G) \cdot \sum_{i=2}^t |V_i|.$$

Note that $|\bar{V}_1| = \sum_{i=2}^t |V_i| \geq t - 1$. We can conclude that

$$|E_{\mathcal{P}}(G)| \geq \frac{1}{2} \delta(G) \cdot (t - 1) + \frac{1}{2} \delta(G) \cdot (t - 1) = \delta(G) \cdot (t - 1).$$

Case 2. $|V_1| < \frac{n}{2}$.

In this case, we consider two subcases, according to the value of t .

Subcase 2.1. $t \geq 2n^{\frac{1}{2}}$.

Let $\mathcal{P}_1 = \{V_i | 1 \leq i \leq t, V_i \text{ contains no small vertex}\}$ and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$. Then we have that

$$|E_{\mathcal{P}}(G)| = \frac{1}{2} \sum_{i=1}^t |E(V_i, \bar{V}_i)| = \frac{1}{2} \left(\sum_{V_i \in \mathcal{P}_1} |E(V_i, \bar{V}_i)| + \sum_{V_j \in \mathcal{P}_2} |E(V_j, \bar{V}_j)| \right).$$

Note that $\delta(G) \leq \log n/30$ a.s. For any $V_i \in \mathcal{P}_1$, by Lemmas 3.3 and 3.4, $|E(V_i, \bar{V}_i)| \geq (\log n/10) \cdot |V_i|$ and for any $V_j \in \mathcal{P}_2$, by Lemmas 3.4 and 3.5, $|E(V_j, \bar{V}_j)| \geq \delta(G) \cdot |V_j|$. Moreover, by Lemma 3.2, $|\mathcal{P}_2| \leq n^{1/2}$. Therefore,

$$\begin{aligned} |E_{\mathcal{P}}(G)| &\geq \frac{1}{2} \left(\frac{\log n}{10} \cdot |\mathcal{P}_1| + \delta(G) \cdot |\mathcal{P}_2| \right) \\ &\geq \frac{1}{2} \left(3\delta(G) \cdot \left(t - n^{\frac{1}{2}} \right) + \delta(G) \cdot n^{\frac{1}{2}} \right) \\ &\geq \frac{1}{2} \left(3\delta(G) \cdot t - 2\delta(G) \cdot n^{\frac{1}{2}} \right) \\ &\geq \delta(G) \cdot t > \delta(G) \cdot (t - 1). \end{aligned}$$

Subcase 2.2. $t < 2n^{\frac{1}{2}}$.

Note that by Lemmas 3.1, 3.4 and 3.5, for any $1 \leq i \leq t$, $|E(V_i, \bar{V}_i)| \geq \delta(G) \cdot |V_i|$ a.s. Then we get that

$$\begin{aligned} |E_{\mathcal{P}}(G)| &= \frac{1}{2} \sum_{i=1}^t |E(V_i, \bar{V}_i)| \geq \frac{1}{2} \delta(G) \cdot \sum_{i=1}^t |V_i| \\ &= \frac{1}{2} \delta(G) \cdot n > 2\delta(G) \cdot n^{\frac{1}{2}} > \delta(G) \cdot (t - 1). \end{aligned}$$

Combining the two cases discussed above, we can conclude that G has $\delta(G)$ edge-disjoint spanning trees. It immediately implies that $\sigma(G) \geq \delta(G)$. We thus complete the proof of Theorem 1.2. ■

4 Proof of Theorem 1.3

Proof of Theorem 1.3: Recall that $G \sim G(n, p)$ with $p \geq 51 \log n/n$. We first bound the minimum degree and the maximum degree of G . Let v be an arbitrary vertex of G . Then $\deg(v)$, the degree of v , obeys the binomial distribution $\text{Bin}(n-1, p)$. By $\mathbf{E}(\deg(v)) = (n-1)p$ and Chernoff's inequality 2.1,

$$\Pr \left(\deg(v) \geq \frac{3}{2}(n-1)p \right) \leq \exp \left(-\frac{(n-1)p}{16} \right) = o(n^{-2}).$$

Hence, by the union bound, with probability at least $1 - o(n^{-1})$, $\Delta(G) \leq \frac{3}{2}(n-1)p$.

On the other hand,

$$\Pr \left(\deg(v) \leq \frac{4}{5}(n-1)p \right) \leq \exp \left(-\frac{(n-1)p}{50} \right) = o(n^{-1.01}).$$

By the union bound again, it follows that with probability at least $1 - n^{-0.01}$, $\delta(G) \geq \frac{4}{5}(n-1)p$. Then we can deduce that a.s.

$$\sigma(G) \leq \frac{|E(G)|}{n-1} \leq \frac{\Delta(G) \cdot n}{2(n-1)} \leq \frac{3}{4}np < \frac{4}{5}(n-1)p \leq \delta(G),$$

the proof is thus completed. ■

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